The Average Connectivity of an Arithmetic Graph

L. Mary Jenitha*

Department of Mathematics,

Manonmaniam Sundaranar University, Tirunelveli.

S. Sujitha

Department of Mathematics,

Holy Cross College (Autonomous), Nagercoil.

E-mail: jsujivenkit@gmail.com

B. Uma Devi

Department of Mathematics,

S.T. Hindu College, Nagercoil.

E-mail: umasub1968@gmailcom

**Corresponding author E-mail: jeni.mathematics@gmail.com*

Abstract

The average connectivity $\bar{\kappa}(G) = \frac{\sum_{u,v} \kappa_G(u,v)}{\binom{\upsilon}{2}}$, $\kappa G(u, v)$ is defined to be the maximum value of k for which u and v are k-connected. In this paper, we consider the concept of the average connectivity of an arithmetic graph. It is shown that $\bar{\kappa}(G) \leq \frac{\left[(\upsilon-2)\binom{\upsilon-\beta}{2}+(\upsilon-\beta)\binom{\beta}{2}+(\upsilon-\beta)^2\beta\right]}{\binom{\upsilon}{2}}$ where ν is the order and β is an independence number of an arithmetic graph. Also, it is clear that, if a_1 is increasing then $\bar{\kappa}(G)$ is decreasing for an arithmetic graph G = Vn, where n $=P_1^{a_1} \times P_2$.

Keywords: Average Connectivity, Arithmetic Graph, Total Connectivity.

AMS subject classification: 05C12

1. Introduction

A *graphG* is an ordered triple$(V(G),E(G),\Psi_G)$ consisting of an nonempty set $V(G)$ of vertices, a set $E(G)$ of edges and an incidence function Ψ_G ,that associates with each edge of G an unordered pair of vertices of G. The number of vertices in G is denoted by$\nu = |V(G)|$ is called the *order* of G while the number of edges in G is denoted by$\varepsilon = |E(G)|$ is called the *size* of the graph G. A graph of order ν and sizeε is called (ν,ε) graph. A graph is *simple* if it has no loops and no two of its links join the same pair of vertices. A simple graph in which each pair of distinct vertices is joined by an edge is called *complete graph.* The *degree* of a vertex v in a graph G is the number of edges of G incident with v and is denoted by $deg_G v$ or $d(v)$.A vertex of degree one is called a *pendent vertex* or an *end vertex* of G. The maximum and minimum degree of a graph G is denoted by Δ (G) and □ (G) respectively.

A vertex v of G is a *cut vertex* if E can be partitioned into E_1 and E_2 such that $G\,[E_1]$ and $G\,[E_2]$ have just the vertex v in common. A *bipartite graph G* is a graph whose vertex set $V\,(G)$ can be partitioned into two subsetsV_1andV_2 such that every edge of G joins V_1 withV_2 ;($V1$, $V2$)is a bipartition of G. A graph G is called *acyclic* if it has no cycles. A connected acyclic graph is called a *tree*. A non trivial path is a tree with exactly two end vertices. A family of paths in G is said to be *internally disjoint* if no vertex of G is

Received: 20 Dec 2018/Accepted: 20 Jan 2019

1553-9105

an internal vertex of more than one path of the family. The *arithmetic graph* $G=V_n$ is introduced by Vasumathi. N and Vangipuram. S in [6] and later it was studied by various authors in [3, 4, 5]. It is defined as, a graph with its vertex set is the set consists of the divisors of n(excluding 1) where n is a positive integer and $n=P_1^{a_1} \times P_2^{a_2} \times P_3^{a_3} \ldots \ldots \ldots \times P_r^{a_r}$ where P_i's are distinct primes and a_i's$\geq$1and two distinct vertices a, b which are not of the same parity are adjacent to this graph if $(a, b)= P_i$ for some i, $1\leq i \leq r$. The vertices a and b are said to be of the same parity if both a and b are the powers of the same prime, for instance $a = P^2$, $b=P^5$.Throughout the paper G is a simple connected graph with at least three vertices. The following observations are used in the sequel.

Observation 1.1.[4]Let $G = V_n$ be an arithmetic graph where $n=P_1^{a_1} \times P_2^{a_2} \times P_3^{a_3} \times \ldots \ldots \ldots \times P_r^{a_r}$ the number of vertices of G is $|V| =[\prod_{i=1}^{r}(a_i + 1)] - 1$.

Observation 1.2.[3]Let $G=V_n$ be an arithmetic graph where $n = P_1^{a_1} \times P_2^{a_2} \times P_3^{a_3} \times \ldots \ldots \ldots \times P_r^{a_r}$,$a_i = 1 \forall i \in \{1,2,3, \ldots r\}$.Then

(1)$\Delta (G) = 2^{r-1}$

(2)□ $(G) =\begin{cases} r, & r \geq 3 \\ 1, & r = 2 \end{cases}$

Observation 1.3.[4]Let $G =V_n$ be an arithmetic graph where $n=P_1^{a_1} \times P_2^{a_2} \times P_3^{a_3} \times \ldots \ldots \ldots \times P_r^{a_r}$, at least one a_i>1.Then $(G) = r$ and $\Delta(G)= a_j \prod_{\substack{i=1 \\ i\neq j}}^{r}(a_i + 1)$-1, where a_j is the maximum exponent of P_i, $i\in \{1,2,3 \ldots r\}$

Observation 1.4.[1]$\sum_{v\in V} d(v) = 2\varepsilon$

2. Average Connectivity

The definition of an average connectivity is studied from [2] and is determined the same for arithmetic graphs.

Definition 2.1.[2]

The *average connectivity* $\bar{\kappa}(G) =\frac{\sum_{u,v} \kappa_G(u,v)}{\binom{v}{2}}$,$\kappa_G(u, v)$is defined to be the maximum value of k for which u and v are k-connected. If the order of$G$ is$v$, then the *average connectivity* $\bar{\kappa}(G) =\frac{\sum_{u,v} \kappa_G(u,v)}{\binom{v}{2}}$, the expression$\sum_{u,v} \kappa_G(u, v)$ is sometimes referred to as the *total connectivity* of G.

Remark 2.2. Maximum number of internally disjoint paths between v_i and v_j are less than or equal to min (deg (v_i), deg (v_j)).

Theorem 2.3. For an arithmetic graph $G=V_n$

(i) $\bar{\kappa}(G) = 1$ if $n=P_1 \times P_2$.

(ii)$\bar{\kappa}(G) < 2^{r-1}$ if $n= P_1 \times P_2 \times P_3 \times \ldots . \times P_r$.

(iii)$\bar{\kappa}(G) < a_j \prod_{\substack{i=1 \\ i\neq j}}^{r}(a_i + 1) - 1$if $n=P_1^{a_1} \times P_2^{a_2} \times P_3^{a_3} \times \ldots \ldots \ldots \times P_r^{a_r}$.

Proof.

(i)In this case, the given arithmetic graph is a non-trivial tree,and hence the result is obvious.

(ii)Let $G =V_n$ be an arithmetic graph where $n=P_1 \times P_2 \times P_3 \times \ldots . \times P_r$.By observation1.1, 1.2 and remark 2.1,and also the arithmetic graph is not regular we get the average connectivity $\bar{\kappa}(G) < 2^{r-1}$

(iii)In this case, the given arithmetic graph $G =V_n$ where $n=P_1^{a_1} \times P_2^{a_2} \times P_3^{a_3} \ldots \ldots \ldots \times P_r^{a_r}$has maximum degree $a_j \prod_{\substack{i=1 \\ i\neq j}}^{r}(a_i + 1)$-1. Since the graph is not regular

Received: 20 Dec 2018/Accepted: 20 Jan 2019

$$\bar{\kappa}(G) < a_j \prod_{\substack{i=1 \\ i \neq j}}^{r} (a_i + 1)\text{-}1.$$

Theorem 2.4. Let $G = V_n$ where $n=P_1 \times P_2 \times P_3 \times \ldots .\times P_r$ be an arithmetic graph of order $\nu = 2^r - 1$ and an independence number β, then $\bar{\kappa}(G) \leq \frac{\left[2^{r-1}\binom{\nu-\beta}{2}+(\nu-\beta)\binom{\beta}{2}+(\nu-\beta)^2\beta\right]}{\binom{\nu}{2}}$.

Proof. Consider the arithmetic graph $G = V_n$ where $n=P_1 \times P_2 \times P_3 \times \ldots .\times P_r$ of order 2^r-1. Let S be the set of independent vertices with $|S|=\beta$.

Since by observation1.1,1.2 and remark 2.1, the average connectivity between any two pair of verticesis at most 2^{r-1}. Therefore, the total connectivity of G is the sum of the following cases, the vertices which are in S and not in S

Case (i) If $u, v \notin S$ then the total connectivity$\sum_{u,v\notin S} \kappa_G(u,v)$is at most $2^{r-1}\binom{\nu-\beta}{2}$.

Case (ii) If $u$or$v$ (or both) is in S then the total connectivity is at most$(\nu-\beta)$.Hence for these pairs$\sum_{u,v} \kappa_G(u,v)$ will be at most $(\nu-\beta)[\binom{\beta}{2} + (\nu-\beta)\beta]$

Hence from these two cases we get $\bar{\kappa}(G) \leq \frac{\left[2^{r-1}\binom{\nu-\beta}{2}+(\nu-\beta)\binom{\beta}{2}+(\nu-\beta)^2\beta\right]}{\binom{\nu}{2}}$.

Theorem 2.5. Let $G = V_n$ be an arithmetic graph of order ν and independence number β, where $n=P_1^{a_1} \times P_2^{a_2} \times P_3^{a_3} \ldots\ldots\ldots \times P_r^{a_r}$ and $n \neq P_1 \times P_2$, then $\bar{\kappa}(G) \leq \frac{\left[(\upsilon-2)\binom{\upsilon-\beta}{2}+(\upsilon-\beta)\binom{\beta}{2}+(\upsilon-\beta)^2\beta\right]}{\binom{\upsilon}{2}}$.

Proof. Let G be an arithmetic graph withνvertices and independence number β, and let S be a set of independent vertices such that $|S| = \beta$. Since the arithmetic graph is not complete, the connectivity between any pair of vertices in G is at mostν -2, so the contribution to the total connectivity of G of the pairs of vertices not in S is bounded by$(\nu-2)\binom{\nu-\beta}{2}$.

On the other hand, if $u$or$v$ (or both) is in S then $k_G(u,v) \leq \nu$ -β, so such pairs contribute at most$(\nu-\beta)\binom{\beta}{2} + (\nu-\beta)^2\beta$ to the total connectivity. Addition of these two contributions gives the desired result.

Corollary 2.6. Let $G = V_n$ be an arithmetic graph where $n=P_1^{a_1} \times P_2^{a_2} \times P_3^{a_3} \ldots\ldots\ldots \times P_r^{a_r}$ at least one a_i>1with order$\nu = [\prod_{i=1}^{r}(a_i + 1)]$-1 and independence number βthen$\bar{\kappa}(G) \leq \frac{\left[[a_j \prod_{\substack{i=1 \\ i\neq j}}^{r}(a_i+1)-1]\binom{\upsilon-\beta}{2}+(\upsilon-\beta)\binom{\beta}{2}+(\upsilon-\beta)^2\beta\right]}{\binom{\upsilon}{2}}$,where a_j is the maximum exponent of P_i.

Proof. The result is obvious from theorem 2.4.

Theorem2.7. For an arithmetic graph $G = V_n$, $n=P_1^{a_1} \times P_2^{a_2}, a_1, a_2 \geq 1$ then$\varepsilon = 4a_1a_2 - a_1 - a_2$, where ε is the size of the graph G.

Proof. The vertex set $V(G)$ contains primes, prime powers, and product of powers. The neighbors of P_1 is a set $N(P_1)$,containing vertices, which are the Cartesian product of the sets $\{P_1, P_1^2, P_1^3 \ldots, P_1^{a_1}\}$ and $\{P_2, P_2^2, P_2^3 \ldots, P_2^{a_2}\}$.Similarly the vertices of N(P_2).

The vertices $P_1^{a_1}, a_1$>1 are adjacent to $P_1 \times P_2$, $P_1 \times P_2^2, \ldots\ldots\ldots$, $P_1 \times P_2^{a_2}$. Also the vertices $P_2^{a_2}, a_2 > 1$ are adjacent to $P_1 \times P_2, P_1^2 \times P_2, \ldots\ldots P_1^{a_1} \times P_2$.Thevertices$P_1^{a_1} \times P_2, a_1$ 1areadjacentto$P_1, P_2, P_2^2, P_2^3 \ldots, P_2^{a_2}$. Similarly, the vertices of the form $P_1 \times P_2^{a_2}, a_2 \geq 1$ are adjacent to$P_1, P_2, P_1^2, P_1^3 \ldots, P_1^{a_1}$. If a_1>1 and $a_2 > 1$ then $P_1^{a_1} \times P_2^{a_2}$ is adjacent only to P_1 and P_2. Hence the degrees of the vertices are given by

Received: 20 Dec 2018/Accepted: 20 Jan 2019
1553-9105 /

$$d(v) = \begin{cases} a_1 a_2 & if \quad v = P_1 or P_2 \\ a_2 & if \quad v = {P_1}^m, 1 < m \leq a_1 \\ a_1 & if \quad v = {P_2}^n, 1 < n \leq a_2 \\ a_1 + a_2 & if \quad v = {P_1}^m \times {P_2}^n; \quad m, n = 1 \\ a_2 + 1 & if \quad v = {P_1}^m \times {P_2}^n; \quad n = 1, 1 < m \leq a_1 \\ a_1 + 1 & if \quad v = {P_1}^m \times {P_2}^n; \quad m = 1, 1 < n \leq a_2 \\ 2 & if \quad v = {P_1}^m \times {P_2}^n, 1 < m \leq a_1, 1 < n \leq a_2 \end{cases}$$

$$\sum d(v) = 2\,a_1 a_2 + (a_1 - 1)a_2 + (a_2 - 1)a_1 + a_1 + a_2 + (a_1 - 1)(a_2 + 1) + (a_2 - 1)(a_1 + 1) + (a_1 - 1)(a_2 - 1)2.$$

$$\sum d(v) = 8\,a_1 a_2 - 2a_1 - 2a_2\,.$$

Therefore by Observation 1.4, we have

$$\boldsymbol{\varepsilon} = \frac{\sum \boldsymbol{d(v)}}{\boldsymbol{2}}$$

$$\boldsymbol{\varepsilon} = 4a_1 a_2 - a_1 - a_2$$

Theorem2.8. For an arithmetic graph $G = V_n$, $n={P_1}^{a_1} \times P_{2,}$ $a_1 > 1$ then $\bar{\kappa}(G) \epsilon [1,2)$,Further if $a_1$is increasing then $\bar{\kappa}(G)$ is decreasing.

Proof. Let $G = V_n$ be an arithmetic graph where $n={P_1}^{a_1} \times P_2$, byobservaion1.1andtheorem2.7, weget the number of pendent vertices are$a_1 - 1$.The contribution to the total connectivity will be reduced if the pendent vertices are increased. Hence $\bar{\kappa}(G)$ is decreasing if $a_1$is increasing.

Theorem2.9. For an arithmetic graph $G = V_n$,$n={P_1}^{a_1} \times {P_2}^{a_2}, a_{1,} a_2 \geq 1$ then G is a bipartite graph.

Proof. Let $G=V_n$be an arithmetic graph, such that $V(G)=X_1 U X_2$,where $X_1=\{p_i, {p_i}^n, 1\leq n \leq a_i\, ; i = 1,2, \ldots\ldots k\}$,$X_2= \{{p_i}^m \times {p_j}^n; 1\leq m,n \leq a_i,\ i=1,2,\ldots.k\}$.By the definition of an arithmetic graph no two vertices of the set $X_1$are adjacent as well as no two vertices of the set X_2 are also adjacent and every edge joins a vertex of X_1 to a vertex of X_2.This shows that the graph G is a bipartite graph.

3. Conclusion

We conclude by noting that the average connectivity of an arithmetic graph is strictly less than the maximum degree of G. Also, for an arithmetic graph $G = V_n$, $n={P_1}^{a_1} \times P_{2,}$ $a_1 > 1$, if $a_1$is increasing then $\bar{\kappa}(G)$ is decreasing and for$G = V_n$, $n={P_1}^{a_1} \times {P_2}^{a_2}, a_{1,} a_2 \geq 1$,$G$ is a bipartite graph. The readers can classify the different arithmetic graphs as in terms of multipartite graphs.

References

[1] J.A. Bondy, U.S.R. Murty, Graph theory with applications, London: Macmillan, 1976.

[2] W. Lowell Beineke, R. Ortrud Oellermann, E. Raymond Pippert, The average connectivity of a graph, Discrete Mathematics, 252, 2002, 31-45.

[3] L. Mary Jenitha, S. Sujitha, The Connectivity Number of an Arithmetic Graph, International journal of Mathematical Combinatorics Special, 1, 2018, 132-136.

[4] R. Rangarajan, A. Alqesmath , A. Alwardi, On V_n- Arithmetic graph ,International Journal of Computer Applications(0975-8887), 125 (9), 2015, 1-7.

[5] K.V. Suryanarayana Rao, V. Sreenivasan, The Split Domination in Arithmetic Graphs, International Journal of Computer Applications (0975-8887), 29 (3), 2011, 46-49.

[6] N .Vasumathi, S. Vangipuram, Existence of a graph with a given domination Parameter, Proceedings of the Fourth Ramanujan Symposium on Algebra and its Applications; University of Madras, Madras, 1995, 187-195.

Received: 20 Dec 2018/Accepted: 20 Jan 2019